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$\mathcal{N}=4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbb{Z}_2$: higher rank case

Masao Jinzenji

Division of Mathematics, Graduate School of Science, Hokkaido University
Sapporo 060-0810, Japan
E-mail: jin@math.sci.hokudai.ac.jp

Toru Sasaki

Department of Physics, Hokkaido University
Sapporo 060-0810, Japan
E-mail: sasaki@particle.sci.hokudai.ac.jp

Abstract: We derive the partition function of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbb{Z}_2$ for gauge group SU(N). We generalize the method of our previous work for the SU(2) case to the SU(N) case. The resulting partition function is represented as the sum of the product of Göttche formula of singular quotient space $T^4/\mathbb{Z}_2$ and of blow-up formulas including A_{N-1} theta series with level N.

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Contents

1. Introduction

$\mathcal{N}=4$ supersymmetric Yang-Mills theory on 4 dimensional manifold has the largest number of supersymmetry if it does not couples to gravity. This theory is one of the well-known examples of the theories that are conformally invariant and finite. In [1], Montonen and Olive conjectured that the theory has a kind of duality under the inverse operation of gauge coupling constant $4\pi i/g^2 \to -g^2/4\pi i$. In [2], Vafa and Witten considered the twisted version of $\mathcal{N}=4$ supersymmetric Yang-Mills theory and generalized the Montonen-Olive duality, by combining the coupling constant with theta angle $\tau := 4\pi i/g^2 + \theta/2\pi$, to the symmetry of the partition function under the subgroup of $\mathrm{SL}(2,\mathbb{Z})$ (S-duality). Using this conjecture as an assumption, they derived the partition function of the theory on a complex surface with ample canonical bundle. The key observation of the derivation is the mass perturbation of $\mathcal{N}=4$ theory down to $\mathcal{N}=1$ theory. This perturbation does not change the partition function. With this operation, the partition function is separated into the following two factors (the same method was also used in [3]). One part is the contribution from bulk (non-zero locus of the section of the canonical bundle), or massive part. This

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contribution is expressed in terms of a kind of Göttsche formula [4, 5, 2], using analogy with the result of $K3$ surface (note that $K3$ surface has trivial canonical bundle). The other part is the contribution from cosmic string (zero locus of the section of the canonical bundle), or massless part. It is determined from the observation of the change of partition function under blow-up of complex surface, originally derived in [6]. Finally, they combined these two factors to satisfy the desired property imposed from the S-duality conjecture. After their work, deeper analyses and further applications were carried out in [7]–[20]. Especially in [15], Sako and T.S. (one of the authors) revealed the fact that Euler number of instanton moduli space and Seiberg-Witten invariants are connected in the framework of Vafa-Witten theory.

In our previous work [9], we derived the partition function of $\mathcal{N}=4$ supersymmetric Yang-Mills theory on orbifold-$T^4/\mathbb{Z}_2$ for SU(2) gauge group from the point of view of orbifold construction. Our idea is very simple. In orbifold construction, we first obtain a quotient space S_0 which has trivial canonical bundle and sixteen double singularities. Next, we blow-up sixteen singularities (they turn into sixteen $\mathcal{O}(-2)$ curves) and obtain a Kummer surface, a special class of $K3$ surfaces. Therefore we speculated that the SU(2) partition function of $K3$ surface should be expressed as the product of the bulk contribution of the quotient space S_0 and of blow-up formulas coming from blowing up the sixteen singularities of S_0. Our speculation was proved to be true in this case [9], with the aid of some identities between Jacobi's theta functions and Dedekind's eta function.

In this paper, we try to generalize this construction to the case of SU(N) gauge group (N is odd prime number). For SU(N) theory it is well-known that the partition function on $K3$ is given by a Hecke transformation of order N of $1/\eta^{\chi(K3)}(\tau)$. Analogous results on gauge group SU(N)$/\mathbb{Z}_N$ for $K3$, $\frac{1}{2}K3$ and T^4 surfaces are also known. This fact was derived and used from physical side in [2, 14, 12], and was confirmed mathematically in [18]. From this starting point, we asked whether $K3$ partition function for SU(N) (or SU(N)$/\mathbb{Z}_N$) can be expressed as the sum of the product of the following two factors: the bulk contribution of $T^4/\mathbb{Z}_2 = S_0$ and $\mathcal{O}(-2)$ curve blow-up formula from 16 orbifold singularities. In this case, our answer turns out to be negative if we use the $\mathcal{O}(-2)$ curve blow-up formulas. But we found that the above construction is possible if we use the $\mathcal{O}(-N)$ curve blow-up formula derived in [10] instead. Here we mean $\mathcal{O}(-N)$ curve blow-up formula by the contribution to the partition function from the $\mathbb{P}^1$ in the surface with self-intersection number $-N$. The formula we obtained is given as follows:

$$Z_k^{K3} = \sum_{j=0}^{N-1} Z_j^{S_0} Z_{k-j+1}^B\,, \tag{1.1}$$

$$Z_j^B = \sum_{\{\beta_l\}} a^j_{\{\beta_l\}} \prod_{l=1}^{16} \frac{\theta^{\beta_l}_{(N)}(\tau)}{\eta^N(\tau)}. \tag{1.2}$$

Here Z_j^X stands for the partition function on X with 't Hooft flux $v \in H^2(X,\mathbb{Z}_N)$, $v^2 = 2j \pmod N$. $\theta^{\beta_l}_{(N)}(\tau)/\eta^N(\tau)$ is the $\mathcal{O}(-N)$ curve blow-up formula labeled by β_l [10]. We emphasize that $a_{\{\beta_l\}}$ is an integer. The key relation in our derivation of (1.2) is the well-known "denominator identity" in the theory of affine Lie algebras. This fact seems to suggest to have a close relation between $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and rational conformal field theory (affine Lie algebra), as was already pointed out in [2].

$\mathcal{O}(-N)$ **curve blow-up.** In this paper, we found the following two key relations. One is given by,

$$\frac{1}{\eta(\frac{\tau}{N})} = \frac{\theta^2_{A_{N-1}}(\tau)}{\eta^N(\tau)}\,, \tag{1.3}$$

where $\theta^2_{A_{N-1}}(\tau)$ is a theta series related to affine Lie algebra. The other is the following:

$$\theta_{A_{N-1}}(\tau) = \sum_\beta a_\beta \theta^\beta_{(N)}(\tau)\,, \tag{1.4}$$

where $\theta^\beta_{(N)}(\tau)$ is Kapranov's theta function labeled by β associated with $\mathcal{O}(-N)$ curve blow-up [10]. These blow-up formulas correspond to the generating functions of the Euler number of "relative moduli space" of $\mathbb{P}^1$ with self intersection number $-N$ in complex surface. By the word "relative" we mean to consider the configuration localized near $\mathbb{P}^1$ like the case of "local mirror symmetry". Here, we again emphasize that the coefficient a_β is an integer. Using these two key relations, we can rewrite $1/\eta(\tau/N)$ by $\mathcal{O}(-N)$ curve blow-up formulas and obtain (1.2). We cannot still figure out the reason why we face $\mathcal{O}(-N)$ curve blow-up formula instead of $\mathcal{O}(-2)$ curve blow-up formula. Hence the geometrical interpretation of our result should be pursued further. The fact that a_β is an integer seems to suggest the possibility of geometrical interpretation. We can easily see that $\mathcal{O}(-N)$ curve blow-up formula appearing in our formulas comes from level N representation of affine SU(N) characters. On the other hand, level k characters of affine SU(N) algebra appears in $\mathcal{N} = 4$ super Yang-Mills theory with gauge group U(k) on A_{N-1} ALE space [2, 21]. From this point of view, the appearance of $\mathcal{O}(-N)$ curve blow-up formula seems to be natural, because we consider SU(N) gauge group [2, 21].

This paper is organized as follows. In section 2, we review our previous work on SU(2) gauge group and derive the bulk contribution of S_0 for gauge group SU$(N)/\mathbb{Z}_N$. In section 3, we introduce the key identity (1.3) and prove it by using a denominator identity of affine Lie algebra. In section 4, we give a key conjecture and rewrite $1/\eta(\tau/N)$ in terms of $\mathcal{O}(-N)$ curve blow-up formulas. In section 5, we derive the partition function of $K3$ for SU$(N)/\mathbb{Z}_N$ by using the results of section 3 and 4. In section 6, we conclude and discuss remaining problems.

2. Review of the SU(2) case and contribution from untwisted sector

In this section, we briefly review the geometrical background of orbifold $T^4/\mathbb{Z}_2$ [22, 23] and derive the contribution from the untwisted sector of $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory on orbifold-$T^4/\mathbb{Z}_2$ for SU(N) [2, 8, 12, 17, 13]. Next, we review our previous work on the reconstruction of $K3$ partition function for SU(2) case, where we use $\mathcal{O}(-2)$ curve blow-up formula [6, 9].

2.1 Orbifold $T^4/\mathbb{Z}_2$

In this subsection, we construct $K3$ surface from T^4 surface by orbifold construction, in keeping canonical bundle $K_X = 0$ trivial [22, 23].

First, we consider T^4 surface. T^4 surface obviously has a trivial canonical bundle $K_{T^4} = 0$, but has nontrivial Picard groups coming from $\dim(H^1(T^4, \mathbb{C})) = 4$. After the $\mathbb{Z}_2$-identification, we obtain a quotient space $T^4/\mathbb{Z}_2 =: S_0$, which has also a trivial canonical bundle and trivial Picard groups, because $H^1(T^4, \mathbb{C})$ is not invariant under the action of $\mathbb{Z}_2$. Note that S_0 has sixteen orbifold singularities. Therefore, we have to resolve these singularities. For this purpose, we blow-up these singularity points and obtain sixteen $\mathcal{O}(-2)$ curves (we call this process $\mathcal{O}(-2)$ curve blow-up). After this process, S_0 turns into a Kummer surface, a special class of compact and smooth $K3$ surfaces.

Here, we write down the change of Euler numbers of surfaces under the process we have just described,

$$\chi(T^4) = 0 \to \chi(S_0) = 8 \to \chi(K3) = 24\,. \tag{2.1}$$

2.2 Contribution from untwisted sector

General structure of Vafa-Witten conjecture. Following [2, 12, 13, 16], we review the general structure of Vafa-Witten conjecture. For twisted $\mathcal{N} = 4$ $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory with 't Hooft flux $v \in H^2(X, \mathbb{Z}_N)$ on X, Vafa and Witten showed that the partition function of this theory is given by the formula:

$$Z_v^X(\tau) := q^{-\frac{N\chi(X)}{24}} \sum_k \chi(\mathcal{N}(v,k)) q^k \qquad (q := \exp(2\pi i \tau))\,, \tag{2.2}$$

where $\mathcal{N}(v,k)$ is the moduli space of anti-self-dual connections associated to $\mathrm{SU}(N)/\mathbb{Z}_N$-principal bundle with 't Hooft flux v and fractional instanton number $k \in \mathbb{Z}/2N$. τ is the gauge coupling constant including theta angle and $\chi(X)$ is Euler number of X. With this result, they conjectured the behavior of the partition functions under the action of $\mathrm{SL}(2,\mathbb{Z})$ on τ. The most important formula of this conjecture is given by,

$$Z_v^X\left(-\frac{1}{\tau}\right) = N^{-\frac{b_2(X)}{2}} \left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} \cdot \sum_{u \in H^2(X, Z_N)} \zeta_N^{u\cdot v} Z_u^X(\tau)\,, \tag{2.3}$$

where $\zeta_N = \exp(2\pi i/N)$.

For later use, we introduce the notation:

$$\begin{aligned} Z^X_{\mathrm{SU}(N)}(\tau) &:= \frac{1}{N} Z_0^X(\tau)\,, \\ Z^X_{\mathrm{SU}(N)/\mathbb{Z}_N}(\tau) &:= \sum_{u \in H^2(X, Z_N)} Z_u^X(\tau)\,. \end{aligned} \tag{2.4}$$

In this notation, we can obtain the following formula from (2.3):

$$Z_0^X\left(-\frac{1}{\tau}\right) = N^{-\frac{b_2(X)}{2}} \left(\frac{\tau}{i}\right)^{-\frac{\chi(X)}{2}} Z^X_{\mathrm{SU}(N)/\mathbb{Z}_N}(\tau)\,. \tag{2.5}$$

This formula is one of the key points on their explicit determination of the form of the partition function of complex surface with ample canonical bundle.

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Partition function of the untwisted sector of S_0. Next, we derive the partition function of the theory on the intermediate quotient space S_0 as a first step of generalization of our previous work. At first, we consider the moduli space $\mathcal{N}(v,k)$ of $X = T^4$. In this case, we can identify the moduli space $\mathcal{N}(v,k)$ with the moduli space $\bar{\mathcal{M}}_H(N,c_1,c_2)$ of rank N stable sheaves E with Chern classes c_1, c_2 ($c_1 = v \bmod N$ and $k = c_2 - (N-1)v^2/2N$) [16, 17]. At this point, we need to note that we restrict N to prime numbers throughout this paper. This condition makes the structure of $\bar{\mathcal{M}}_H(N,c_1,c_2)$ tractable.

According to [17], the moduli space of rank N (N: prime) stable sheaves E of V is given as follows. First, We introduce Mukai vector in $\oplus_j H^{2j}(X,\mathbb{Z})$,

$$V = ch(E)\sqrt{td_X} = N + c_1 + \frac{c_1^2 - 2c_2}{2}\,. \tag{2.6}$$

We remark here that in our case $X = T^4$, $td_X = 1$. Then we introduce the inner product of Mukai vector,

$$\begin{aligned}\langle V^2\rangle &= -\int_X \left(N + c_1 + \frac{c_1^2 - 2c_2}{2}\right) \vee \left(N + c_1 + \frac{c_1^2 - 2c_2}{2}\right)\\ &= 2Nc_2 - (N-1)c_1^2\,.\end{aligned} \tag{2.7}$$

Here we use a symmetric bilinear form on $\oplus_j H^{2j}(X,\mathbb{Z})$:

$$\begin{aligned}\langle x, y\rangle &= -\int_X (x \vee y)\\ &= \int_X (x_1 y_1 - x_0 y_2 - x_2 y_0)\,,\end{aligned} \tag{2.8}$$

where $x = x_0 + x_1 + x_2, x_j \in H^{2j}(X,\mathbb{Z})$ and $x\vee = x_0 - x_1 + x_2$.

With these preparation, we can describe the moduli space $\bar{\mathcal{M}}_H^X(c_1,c_2)$ explicitly in terms of Hilbert scheme of n points $X^{[n]}$ of X,

$$\bar{\mathcal{M}}_H^X(c_1,c_2) \cong \hat{X} \times (X)^{[\frac{\langle V^2\rangle}{2}]} = \hat{X} \times (X)^{[Nc_2 - \frac{(N-1)c_1^2}{2}]}\,, \tag{2.9}$$

where H is polarization of X, and $\hat{X}$ is the T^4 surface dual to X, which represents non-trivial Picard group [17]. Here, we have to notice that $\bar{\mathcal{M}}_H^{K3}(c_1,c_2)$ is also diffeomorphic to $(K3)^{[\frac{\langle V^2\rangle}{2}+1]}$.

Since S_0 has the trivial canonical bundle like $K3$ and T^4, it doesn't have cosmic strings, which are given by zero locus of the section of the canonical bundle. From the point of view of [2], this fact leads us to expect that the corresponding moduli space of S_0 is also diffeomorphic to Hilbert scheme of points on S_0,

$$\bar{\mathcal{M}}_H(c_1,c_2) \cong (S_0)^{[\frac{\langle V^2\rangle}{2}]} = (S_0)^{[Nn - \frac{(N-1)v^2}{2}]}, \tag{2.10}$$

where we used the fact that the Picard group of S_0 is trivial.[1] Using this assumption and Göttsche formula, we can evaluate the partition function of S_0, i.e., the contribution from

[1]Precisely speaking, we have to take care of non-trivial Todd class of S_0. But we don't know the precise definition of Todd class of S_0. Moreover, this correction does not affect the results of computation severely. Therefore, we neglect here this correction.

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the untwisted sector with $v^2 = 2j \pmod N$ type,

$$\begin{aligned} Z_j^{S_0}(\tau) &= q^{-\frac{1}{3N}} \sum_{v^2 \equiv j \,(\mathrm{mod}\, N), n} e(\bar{\mathcal{M}}_H(V)) q^{n-\frac{N-1}{N}j} \\ &= q^{-\frac{1}{3N}} \sum_n e((S_0)^{[Nn-(N-1)j]}) q^{n-\frac{N-1}{N}j} \\ &= q^{-\frac{1}{3N}} \sum_m e((S_0)^{[m]}) \frac{(q^{\frac{1}{N}})^m + \zeta_N^{-j}(\zeta_N q^{\frac{1}{N}})^m + \cdots + \zeta_N^{-j(N-1)}(\zeta_N^{N-1} q^{\frac{1}{N}})^m}{N} \\ &= \frac{1}{N}\left(\frac{1}{\eta^8(\frac{\tau}{N})} + \zeta_N^{\frac{1}{3}} \zeta_N^{-j} \frac{1}{\eta^8(\frac{\tau+1}{N})} + \cdots + \zeta_N^{\frac{N-1}{3}} \zeta_N^{-j(N-1)} \frac{1}{\eta^8(\frac{\tau+N-1}{N})} \right). \end{aligned} \tag{2.11}$$

For trivial type $v = 0$, which corresponds to SU(N) gauge group, we follow the discussion in [2] and set,

$$Z_t^{S_0}(\tau) = C \frac{1}{\eta^8(N\tau)} + \frac{1}{N}\frac{1}{\eta^8(\frac{\tau}{N})} + \frac{1}{N}\zeta_N^{1/3} \frac{1}{\eta^8(\frac{\tau+1}{N})} + \cdots + \frac{1}{N}\zeta_N^{\frac{N-1}{3}} \frac{1}{\eta^8(\frac{\tau+N-1}{N})}, \tag{2.12}$$

where C is some unknown constant. Note that the term $1/\eta^8(N\tau)$ is obtained from $1/\eta^8(\tau/N)$ by using the modular transformation $\tau \to -1/\tau$, and should be included to satisfy the conjecture (2.5).

2.3 $\mathcal{O}(-2)$ curve blow-up formula and $K3$ partition function: review of the SU(2) case

In this subsection, we review the discussion of our previous work on SU(2) case [9]. Especially, we reconstruct the $K3$ partition function by combining the contribution from the untwisted sector (i.e., S_0) for the SU(2) case with the contribution of blowing up sixteen double singularities of S_0. In [9], we introduced $\mathcal{O}(-2)$ curve blow-up formulas as the contribution from the twisted sector:

$$\frac{\theta_2(\tau)}{\eta(\tau)^2}, \qquad \frac{\theta_3(\tau)}{\eta(\tau)^2}, \qquad \frac{\theta_4(\tau)}{\eta(\tau)^2}. \tag{2.13}$$

They describe the contribution to the partition function coming from $\mathbb{P}^1$ with self intersection number -2. This $\mathbb{P}^1$ appear as the result of blowing up the double singularity in the complex surface. To be precise, we rearrange better the above formulas in the following way,

$$\begin{aligned} \frac{1}{2}\frac{\theta_2(\tau)}{\eta(\tau)^2} &= \frac{1}{2}\frac{\left(\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2}\right)}{\eta(\tau)^2} = \frac{\left(\sum_{n\in\mathbb{Z}} q^{2(n-\frac{1}{4})^2}\right)}{\eta(\tau)^2}, \\ \frac{1}{2}\frac{\theta_3(\tau)+\theta_4(\tau)}{\eta(\tau)^2} &= \frac{\left(\sum_{n\in\mathbb{Z}} q^{2n^2}\right)}{\eta(\tau)^2}, \\ \frac{1}{2}\frac{\theta_3(\tau)-\theta_4(\tau)}{\eta(\tau)^2} &= \frac{\left(\sum_{n\in\mathbb{Z}} q^{2(n-\frac{1}{2})^2}\right)}{\eta(\tau)^2}. \end{aligned} \tag{2.14}$$

Heuristically, these blow-up formulas correspond to the contributions from the rank 2 vector bundles localized near $\mathbb{P}^1$: $\mathcal{O}(nE)\oplus\mathcal{O}((-n+\frac{1}{2})E)$, $\mathcal{O}(nE)\oplus\mathcal{O}(-nE)$, $\mathcal{O}(nE)\oplus\mathcal{O}((-n+1)E)$ respectively. Here, we denote by E $(E\cdot E=-2)$, the divisor corresponding to the above $\mathbb{P}^1$. Notice that the second Chern classes of these vector bundles are given by $2n(n-1/2)$, $2n^2$, $2n(n-1)$ respectively. These numbers coincide with the powers of q in theta series except for constants in n. We will discuss such correspondence more generally in the next section. The factor $1/\eta(\tau)^2$ comes from the boundary part of the moduli space of rank 2 stable sheaves localized near $\mathbb{P}^1$. The rank two stable sheaves E_b on the boundary of the moduli space are written as direct sum of two ideal sheaves I_{Z_1}, I_{Z_2} of Hilbert scheme of points on the normal bundle $N(\mathbb{P}^1)$ of the $\mathbb{P}^1$. Since the total space of $N(\mathbb{P}^1)$ has only 1 2-cycle $\mathbb{P}^1$ as the nontrivial cycles (0-cycle is already counted in the contribution from S_0), it is obvious from Göttsche formula that the corresponding contribution is given by $1/\eta(\tau)^2$. For mathematically rigorous discussion, see [6, 13].

Now we separate 16 double singularities into two groups that consist of 8 double singularities. Then we blow-up 8 singularities in each group at a time. The resulting contributions of these operations are given by the following formulas:

$$\begin{aligned}\tilde{Z}_1(\tau) &= \frac{\theta_2^8(\tau)(\theta_3^8(\tau)+\theta_4^8(\tau))}{2^9\eta^{32}(\tau)}\,,\\ \tilde{Z}_2(\tau) &= \frac{\theta_2^8(\tau)(\theta_3^8(\tau)-\theta_4^8(\tau))}{2^9\eta^{32}(\tau)}\,,\\ \tilde{Z}_3(\tau) &= \frac{\theta_3^8(\tau)\theta_4^8(\tau)}{\eta^{32}(\tau)}\,.\end{aligned} \tag{2.15}$$

In the derivation of $\tilde{Z}_3(\tau)$, we used the Vafa-Witten conjecture on modular property of the partition function. Using these formulas, we reconstructed $K3$ partition function:

$$\begin{aligned}Z^{K3}_{\text{even}}(\tau) &= 4\left(Z^{S_0}_{\text{odd}}(\tau)\tilde{Z}_2(\tau)+Z^{S_0}_{\text{even}}(\tau)\tilde{Z}_1(\tau)\right),\\ Z^{K3}_{\text{odd}}(\tau) &= 4\left(Z^{S_0}_{\text{odd}}(\tau)\tilde{Z}_1(\tau)+Z^{S_0}_{\text{even}}(\tau)\tilde{Z}_2(\tau)\right).\end{aligned} \tag{2.16}$$

$$Z^{K3}_t(\tau) = \frac{1}{4}\frac{1}{\eta^8(2\tau)}\tilde{Z}_3(\tau)+4\left(Z^{S_0}_{\text{even}}(\tau)\tilde{Z}_2(\tau)+Z^{S_0}_{\text{odd}}(\tau)\tilde{Z}_1(\tau)\right). \tag{2.17}$$

To compare these results with those of Vafa-Witten, we rewrite these partition functions using identities between Dedekind's eta function and Jacobi's theta functions:

$$\frac{\theta_3^8(\tau)\theta_4^8(\tau)}{\eta^{32}(\tau)}=\frac{1}{\eta^{16}(2\tau)}\,,\qquad \frac{\theta_2^8(\tau)\theta_3^8(\tau)}{2^8\eta^{32}(\tau)}=\frac{1}{\eta^{16}(\frac{\tau}{2})}\,,\qquad \frac{\theta_2^8(\tau)\theta_4^8(\tau)}{2^8\eta^{32}(\tau)}=e^{\frac{8\pi i}{3}}\frac{1}{\eta^{16}(\frac{\tau+1}{2})}\,. \tag{2.18}$$

Then we obtain the well-known results in [2] as expected,

$$\begin{aligned}Z^{K3}_t(\tau) &= \frac{1}{4}\frac{1}{\eta^{24}(2\tau)}+\frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2})}+\frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2}+\frac{1}{2})}\,,\\ Z^{K3}_{\text{even}}(\tau) &= \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2})}+\frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2}+\frac{1}{2})}\,,\\ Z^{K3}_{\text{odd}}(\tau) &= \frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2})}-\frac{1}{2}\frac{1}{\eta^{24}(\frac{\tau}{2}+\frac{1}{2})}\,.\end{aligned} \tag{2.19}$$

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3. From bulk to blow-up

In the rest of this paper, we generalize our previous work reviewed in section 2 to the SU(N) case. Especially in the following two sections, we introduce the blow-up formula that appears in our theory. In this section, we first derive the blow-up formula in the simplest case, and then introduce key identities which motivated us to generalize the identities in (2.18). Finally, we give a proof of these identities with the aid of the denominator identity of affine Lie algebra [24, 25].

3.1 $\mathcal{O}(-d)$ blow-up formula for the $\mathcal{N}=4$ SU(N) gauge theory

In this subsection, we sketch the derivation of $\mathcal{O}(-d)$ blow-up formula for the $\mathcal{N}=4$ SU(N) gauge theory. Roughly speaking, the blow-up formula is the generating function of Euler number of the moduli space of the rank N stable sheaves localized near $\mathbb{P}^1$ in a complex surface with self-intersection number $-d$. Heuristically, the bulk part of the moduli space is separated into connected components labeled by the following rank N vector bundle on a complex surface,

$$\begin{aligned} &\mathcal{O}(m_1 E_d) \oplus \mathcal{O}(m_2 E_d) \oplus \cdots \oplus \mathcal{O}(m_N E_d)\,, \\ &m_i \in \mathbb{Z}\,, \qquad m_1 + m_2 + \cdots + m_N = 0\,, \end{aligned} \tag{3.1}$$

where E_d ($E_d \cdot E_d = -d$) stands for the divisor corresponding to the above $\mathbb{P}^1$. The condition $m_1 + m_2 + \cdots + m_N = 0$ is imposed by the vanishing first Chern class. The second Chern class of the bundle in (3.1) is given by,

$$\left(\sum_{i<j} m_i m_j\right) E_d \cdot E_d = \frac{d}{2}\sum_{i=1}^{N}(m_i)^2, \tag{3.2}$$

where we used the condition $m_1+m_2+\cdots+m_N=0$. But the lattice $\{(m_1,m_2,\cdots,m_N) \in \mathbb{Z}^N |\ m_1+m_2+\cdots+m_N=0\}$ with usual Euclid metric in $\mathbb{R}^N$ is nothing but the A_{N-1} root lattice. Therefore, the connected components in (3.1) are labeled by a root vector $n_1\alpha_1 + \cdots + n_{N-1}\alpha_{N-1}$. The second Chern class in (3.2) turns out to be $\frac{d}{2}({}^t n A_{N-1} n)$. Here, we denote by A_{N-1} Cartan matrix of A_{N-1} root lattice. It can be shown that Euler number of each connected component is equal to 1 [10]. Then it follows that the contribution from the bulk part of the moduli space is given by the theta series:

$$\theta^{0,d}_{A_{N-1}}(\tau) := \sum_{m\in\mathbb{Z}^{N-1}} q^{\frac{d}{2}({}^t m A_{N-1} m)}\,. \tag{3.3}$$

Next, we consider the contribution from the boundary part of the moduli space. As we have discussed in the previous section, the rank N stable sheaves in the boundary parts of the moduli space are described by direct sum of N ideal sheaves $I_{Z_1}, I_{Z_2}, \cdots, I_{Z_N}$ of Hilbert scheme of points on $N(\mathbb{P}^1)$. Therefore, we can conclude that the contribution to the generating function is given by $1/\eta(\tau)^N$, using the same discussion as in the SU(2) case. Combining the two contributions, we are led to the $\mathcal{O}(-d)$ blow-up formula for $\mathcal{N}=4$ SU(N) gauge theory:

$$\frac{\theta^{0,d}_{A_{N-1}}(\tau)}{\eta(\tau)^N} = \frac{\sum_{m\in\mathbb{Z}^{N-1}} q^{\frac{d}{2}({}^t m A_{N-1} m)}}{\left(q^{\frac{N}{24}}\prod_{n=1}^{\infty}(1-q^n)^N\right)}. \tag{3.4}$$

Now, we are at the most important point of generalization of our previous work on $\mathcal{N}=4$ SU(2) gauge theory. Using the same method in the previous section, it can be easily seen that the partition function of $\mathcal{N}=4$ SU$(N)/\mathbb{Z}_N$ gauge theory on $K3$ surface consists of the functions $1/\eta(\frac{\tau+j}{N})$ $(j=0,1,\ldots,N-1)$. Therefore, we have to search for some identities that relate $1/\eta(\frac{\tau+j}{N})$ to $\Theta_{A_{N-1}}(\tau)/\eta(\tau)^N$ (we denote by $\Theta_{A_{N-1}}(\tau)$ some variant of $\theta^{0,d}_{A_{N-1}}(\tau)$). Fortunately, we found the desired identities. Let us discuss this point in the following subsections.

3.2 Key identity

In this subsection, we introduce the identities that represent the functions $1/\eta(N\tau)$ and $1/\eta(\tau/N)$ in terms of the function $\Theta_{A_{N-1}}(\tau)/\eta^N(\tau)$. In the SU(2) case, this identity is simply given by,

$$\frac{1}{\eta(2\tau)} = \frac{\theta_4(2\tau)}{\eta^2(\tau)}, \tag{3.5}$$

or

$$\frac{1}{\eta(\tau/2)} = \frac{1}{2}\frac{\theta_2(\tau/2)}{\eta^2(\tau)}. \tag{3.6}$$

Note that the right hand sides of these identities have the form $\Theta_{A_1}(\tau)/\eta^2(\tau)$. Then we searched for a possible generalization of these identities to the case of SU(N) gauge group. After some trial and error, we found the following key identities:

$$\frac{1}{\eta(N\tau)} = \frac{\theta^1_{A_{N-1}}(\tau)}{\eta^N(\tau)}, \tag{3.7}$$

or alternatively

$$\frac{1}{\eta(\frac{\tau}{N})} = \frac{\theta^2_{A_{N-1}}(\tau)}{\eta^N(\tau)}, \tag{3.8}$$

where we introduced A_{N-1} theta functions,

$$\theta^0_{A_{N-1}}(\tau) := \sum_{m\in\mathbb{Z}^{N-1}} q^{\frac{1}{2}{}^t m A_{N-1} m}, \tag{3.9}$$

$$\theta^1_{A_{N-1}}(\tau) := \sum_{m\in\mathbb{Z}^{N-1}} q^{\frac{1}{2}{}^t m A_{N-1} m} e^{2\pi i m\cdot\delta}, \tag{3.10}$$

$$\theta^2_{A_{N-1}}(\tau) := \sum_{m\in\mathbb{Z}^{N-1}} q^{\frac{1}{2}{}^t (m+\frac{\rho}{N}) A_{N-1} (m+\frac{\rho}{N})}. \tag{3.11}$$

Here, we introduced the vector ρ in $\mathbb{Z}^{N-1}\otimes_{\mathbb{Z}}\mathbb{Q}$:

$$\delta := \frac{1}{N}(1,\ldots,1), \qquad \rho = N A^{-1}_{N-1}\delta, \tag{3.12}$$

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which is the same as the usual ρ given by a half of the sum of the positive roots of Lie algebra A_{N-1}. Note that we can obtain (3.7) by applying the modular transformation $\tau \to -1/\tau$ to (3.8).

3.3 Denominator (Macdonald) identity and affine Lie algebra

In this subsection, we give the proof of the identity in (3.8). As a warming-up, we consider the SU(2) case given in (3.6). In this case, it is well-known that $\theta_2(\tau/2)$ has the following product formula (which follows from Jacobi's triple product identity):

$$\theta_2\left(\frac{\tau}{2}\right) = 2q^{\frac{1}{16}} \prod_{n=1}^{\infty} \left(1 - q^{\frac{n}{2}}\right) \left(1 + q^{\frac{n}{2}}\right)^2 = 2\frac{\eta^2(\tau)}{\eta(\tau/2)}\,. \tag{3.13}$$

Hence the proof of (3.6) is completed.

Next, we turn into the SU(N) case given in (3.8). In this case, We already have the celebrated denominator identity of affine Lie algebra, that corresponds to higher rank version of Jacobi's triple product identity [25]:

Theorem 1. (Denominator Identity)

$$\prod_{\alpha\in\Delta_+} (1 - e(-\alpha))^{\mathrm{mult}\ \alpha} = \sum_{w\in W} \epsilon(w) e(w(\rho) - \rho)\,. \tag{3.14}$$

For convenience of readers, we introduce here the notations of the affine Lie algebras.

Notation 1. *Notations [25, 24],*
Δ : *set of roots,* $\Delta_+ \subset \Delta$: *set of positive roots,* l : *rank of Cartan matrix*
mult α : *multiplicity of* $\alpha \in \Delta_+$,
W : *Weyl Group,* $w \in W$: *its element,* $\epsilon(w) = (-1)^{l(w)}$, $l(w)$: *length of* w,
ρ : *half of the sum of the positive roots,*
$h^\vee$: *dual Coxeter number.*

From now on, the symbol "prime ($'$)" denotes restriction to classical Lie algebra associated with affine Lie algebra.

Since Weyl group W of affine Lie algebra is given by a semi-direct product of classical Weyl group W' and of classical root lattice L, we can rewrite (3.14) into the following form:

Corollary 1.

$$q^{\frac{|\rho|^2}{2h^\vee}} \prod_{n\geq 1} \left((1 - q^n)^l \prod_{\alpha\in\Delta'} (1 - q^n e(\alpha)) \right) = \sum_{\alpha\in L} \chi'(h^\vee \alpha) q^{\frac{|\rho + h^\vee \alpha|^2}{2h^\vee}}\,, \tag{3.15}$$

where

$$\chi'(\lambda) = \frac{\sum_{w\in W'} \epsilon(w) e(w(\lambda+\rho) - \rho)}{\prod_{\alpha\in\Delta'_+} (1 - e(-\alpha))}\,. \tag{3.16}$$

Note that q stands for $e(\alpha_0)$ (α_0 is the null root of affine root system) and that $|u|^2$ is the square length of a vector u in L. In our case, $|\sum_{j=1}^{N-1} m_j\alpha_j|^2 = {}^t m A_{N-1} m$. With these settings, we can prove the identity in (3.8).

Proof of (3.8). Let us apply the formula (3.15) to the affine A_{N-1} Lie algebra. In this case, we have

$$h^\vee = N\,, \qquad l = N-1\,, \qquad \frac{|\rho|^2}{2h^\vee} = \frac{N^2-1}{24}\,. \tag{3.17}$$

Next, we specialize the variable $e(\alpha)$ as follows:

$$e(\alpha) = \exp\left(\frac{2\pi i\alpha\cdot\rho}{N}\right). \tag{3.18}$$

Under this specialization, $\chi'(N\alpha)$ turns into 1 as a consequence of the denominator identity applied to the classical Lie algebra A_{N-1}. Therefore, the identity in Corollary 1 turns into an identity between the following two q-series:

$$\begin{aligned} q^{\frac{N^2-1}{24}} \prod_{n=1}^{\infty}(1-q^n)^{N-1} \prod_{j=1}^{N-1}(1-q^n\zeta_N^j)^N &= \\ &= \sum_{m\in(N\mathbb{Z})^{N-1}} q^{\frac{1}{2N}{}^t(m+\rho)A_{N-1}(m+\rho)}\,. \end{aligned} \tag{3.19}$$

Substituting $q^{1/N}$ into q, we obtain the desired formula (3.8). $\Box$

Note that this proof was first given by Macdonald (p. 120,121) [24].

We have thus shown that the function $1/\eta(\tau/N)$ can be described by $\theta^2_{A_{N-1}}(\tau)/\eta^N(\tau)$ using the key identity. But this identity is not enough to rewrite the function $1/\eta(\tau/N)$ in terms of the blow-up formula, because $\theta^2_{A_{N-1}}(\tau)$ contains all the integer powers of $q^{1/N}$. In the next section, we complete our attempt to rewrite $1/\eta(\tau/N)$ in terms of the blow-up formula of $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory. However, we have shown that this theory is closely related to the affine Lie algebras, as was already pointed out in [2].

4. $\mathcal{O}(-N)$ curve blow-up

We continue the discussion on the blow-up formula of $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory. Especially in this section, we try to rewrite the theta function $\theta^2_{A_{N-1}}(\tau)$ in the previous section as the sum of the $\mathcal{O}(-N)$ curve blow-up formulas of $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory [9, 10]. First, we introduce the key conjecture which we obtained after some trial and error. Next, we show some explicit examples of this conjecture.

4.1 Key conjecture

In the previous section, we obtained an identity that relates $1/\eta(\tau/N)$ to the similar function in (3.4). However in the geometrical context, theta function $\theta^2_{A_{N-1}}(\tau)$ is not the expected theta function appearing in the blow-up formula. In the $N=2$ case, we fortunately have the following duplication formula:

$$\theta^2_{A_1}(\tau)^2 = \frac{1}{2}\theta_2(\tau)\theta_3(\tau)\,. \tag{4.1}$$

That is why we have introduced (2.14) directly instead of $\theta^2_{A_1}(\tau)/\eta^2(\tau)$. As we have already mentioned, relations in (2.14) are $\mathcal{O}(-2)$ curve blow-up formulas of $\mathrm{SU}(2)/\mathbb{Z}_2$ gauge theory and they are consistent with geometrical context (recall section 2.1). In this principle, we first tried to find similar identities for general N case as (4.1), where we can rewrite $\theta^2_{A_{N-1}}(\tau)$ in terms of the theta function appearing in $\mathcal{O}(-2)$ curve blow-up formula. For this purpose, we refer here the theorem of Kapranov on the general form of $\mathcal{O}(-d)$ blow-up formulas of $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory [10]:

Theorem 2. (Kapranov) *$\mathcal{O}(-d)$-curve blow-up formula for $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory is given by the A_{N-1} theta series with level d:*

$$\sum_{a\in L} q^{\Psi(a,f)-d\Psi(a,a)/2}\,, \tag{4.2}$$

where L is the A_{N-1} root lattice, f is an element of the weight lattice and $\Psi(a,b) = -{}^t a A_{N-1} b$.

Remark 1. *In Theorem 2, the factor coming from boundaries of the moduli space is neglected because Kapranov treated uncompactified case in [10]. Compactified version of $\mathcal{O}(-1)$ curve blow-up formula for $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory was first derived by Yoshioka [19].*

Remark 2. *The original version of Theorem 2 in [10] takes f as an element of the root lattice. But in [2] where $\mathcal{O}(-1)$-curve blow-up formula for $\mathrm{SU}(2)/\mathbb{Z}_2$ gauge theory is considered, Vafa and Witten introduced the two types of blow-up formula:*

$$\Theta_0 := \sum_{n\in\mathbb{Z}} q^{n^2}\,, \qquad \Theta_1 := \sum_{n\in\mathbb{Z}} q^{(n-\frac{1}{2})^2}\,. \tag{4.3}$$

Θ_0 and Θ_1 correspond to taking f as 0 and f as $\frac{1}{2}\alpha$ respectively. Therefore, it is natural to take f as an element of the weight lattice. This generalization is also compatible with our treatment of O(-2)-curve blow-up formula for $\mathrm{SU}(2)/\mathbb{Z}_2$ gauge theory.

Then we applied Theorem 2 to the $d = 2$ case and tried to find a formula which represents $\theta^2_{A_{N-1}}(\tau)$ in terms of Kapranov's theta functions. Unfortunately, our attempt failed for some lower N's. But surprisingly enough, we have found some beautiful formulas that represent $\theta^2_{A_{N-1}}(\tau)$ in terms of Kapranov's theta functions with $d = N$:

Conjecture 1. *For odd $N \geq 3$, we can express $\theta^2_{A_{N-1}}(\tau)$ as a linear combination of Kapranov's theta functions with $d = N$:*

$$\theta^2_{A_{N-1}}(\tau) = a_1\theta^{\beta_1}_{(N)}(\tau) + a_2\theta^{\beta_2}_{(N)}(\tau) + \cdots + a_k\theta^{\beta_k}_{(N)}(\tau)\,, \tag{4.4}$$

where

$$\theta^{\beta_j}_{(N)}(\tau) := \sum_{m\in\mathbb{Z}^{N-1}} q^{\frac{N}{2}{}^t(m+v(\beta_j))A_{N-1}(m+v(\beta_j))}, \tag{4.5}$$

$$v(\beta_j) := \frac{1}{N}A^{-1}_{N-1}\begin{pmatrix}\beta^{(1)}_j\\ \beta^{(2)}_j\\ \vdots\\ \beta^{(N-1)}_j\end{pmatrix}, \qquad \beta^{(n)}_j \in \mathbb{Z}\,. \tag{4.6}$$

Moreover, a_j is a non-negative integer.

Using this conjecture, we can describe $\theta^2_{A_{N-1}}(\tau)$ by Kapranov-type theta function associated with $\mathcal{O}(-N)$ curve blow-up formula. Here we briefly explain the reason why we call $\theta^{\beta_j}_{(N)}(\tau)$ Kapranov-type. Using the notation of Kapranov, $\theta^{\beta_j}_{(N)}(\tau)$ can be expressed as,

$$\theta^{\beta_j}_{(N)}(\tau) = \sum_{a\in L} q^{-\Psi(f,f)/2N+\Psi(a,f)-N\Psi(a,a)/2} \qquad (f = N\cdot v(\beta_j))\,, \tag{4.7}$$

and this expression is the same theta function as the one in (4.2) with $d = N$, except for the factor $q^{-\Psi(f,f)/2N}$ independent of $a \in L$. Notice here that Kapranov-type theta function is nothing but the level N theta function [25].

In summary, we have found that the r.h.s. of (3.8) is the sum of the $\mathcal{O}(-N)$ curve blow-up formulas. Appearance of $\mathcal{O}(-N)$ curve blow-up formulas instead of $\mathcal{O}(-2)$ curve blow-up formulas seems to be natural from the point of view of the results of Nakajima [21, 2], that suggests the appearance of level N representation of affine Lie algebra in $\mathcal{N} = 4$ SYM theory with gauge group U(N) on ALE spaces.

4.2 Examples: $N = 3, 5, 7$ case

In this subsection, we give some explicit examples of Conjecture 1, found by use of Maple V.

$N = 3$ case.

$$\theta^2_{A_2}(\tau) = \theta^{(1,0)}_{(3)}(\tau) + \theta^{(2,0)}_{(3)}(\tau) + \theta^{(4,0)}_{i(3)}(\tau)\,, \tag{4.8}$$

where

$$\theta^{(k,l)}_{(3)}(\tau) = \sum_{m\in\mathbb{Z}^2} q^{\frac{3}{2}{}^t(m+v(k,l))A_2(m+v(k,l))}, \tag{4.9}$$

$$v(k,l) = \frac{1}{3}A_2^{-1}\begin{pmatrix} k \\ l \end{pmatrix}. \tag{4.10}$$

In the following, we give explicit expressions of the theta functions $\theta^{(1,0)}_{(3)}(\tau)$, $\theta^{(2,0)}_{(3)}(\tau)$, $\theta^{(4,0)}_{(3)}(\tau)$:

$$\theta^{(1,0)}_{(3)}(\tau) = \sum_{(m_1,m_2)\in\mathbb{Z}^2} q^{\frac{1}{9}+m_1+3(m_1^2+m_2^2-m_1m_2)} =: q^{\frac{1}{9}}\tilde{\theta}^0_{(3)}(\tau)\,, \tag{4.11}$$

$$\theta^{(2,0)}_{(3)}(\tau) = \sum_{(m_1,m_2)\in\mathbb{Z}^2} q^{\frac{4}{9}+2m_1+3(m_1^2+m_2^2-m_1m_2)} =: q^{\frac{4}{9}}\tilde{\theta}^1_{(3)}(\tau)\,, \tag{4.12}$$

$$\theta^{(4,0)}_{(3)}(\tau) = \sum_{(m_1,m_2)\in\mathbb{Z}^2} q^{\frac{16}{9}+4m_1+3(m_1^2+m_2^2-m_1m_2)} =: q^{\frac{7}{9}}\tilde{\theta}^2_{(3)}(\tau)\,. \tag{4.13}$$

For later use, we introduced $\tilde{\theta}^j_{(3)}(\tau)$, $(j = 0, 1, 2)$, which have integral q expansions. We can easily see that the right hand side of (4.10) consists of 3 types of theta functions, which are classified by top q powers modulo $\mathbb{Z}$. We expect that this is the general property of Conjecture 1 for odd N, which was confirmed by computer check in the $N = 5, 7$ cases.

$N = 5$ **case.**

$$
\begin{aligned}
\theta^2_{A_4}(\tau) \;=\;& \left. 5\theta^{(1,1,1,1)}_{(5)}(\tau) + \right\} =: q\tilde{\theta}^4_{(5)}(\tau) \\
&\left. + \theta^{(1,0,0,1)}_{(5)}(\tau) + 2\theta^{(3,1,0,0)}_{(5)}(\tau) + 2\theta^{(1,3,1,0)}_{(5)}(\tau) + \right\} =: q^{\frac{1}{5}}\tilde{\theta}^0_{(5)}(\tau) \\
&\left. + \theta^{(2,0,0,2)}_{(5)}(\tau) + 2\theta^{(6,2,0,0)}_{(5)}(\tau) + 2\theta^{(2,6,2,0)}_{(5)}(\tau) + \right\} =: q^{\frac{4}{5}}\tilde{\theta}^3_{(5)}(\tau) \\
&\left. + \theta^{(0,1,1,0)}_{(5)}(\tau) + 2\theta^{(3,0,1,1)}_{(5)}(\tau) + 2\theta^{(1,0,3,0)}_{(5)}(\tau) + \right\} =: q^{\frac{2}{5}}\tilde{\theta}^1_{(5)}(\tau) \\
&\left. + \theta^{(0,2,2,0)}_{(5)}(\tau) + 2\theta^{(6,0,2,2)}_{(5)}(\tau) + 2\theta^{(2,0,6,0)}_{(5)}(\tau) + \right\} =: q^{\frac{3}{5}}\tilde{\theta}^2_{(5)}(\tau)\,, \qquad (4.14)
\end{aligned}
$$

where

$$
\theta^{(n,k,l,p)}_{(5)}(\tau) \;=\; \sum_m q^{\frac{5}{2}{}^t(m+v(n,k,l,p))A_4(m+v(n,k,l,p))}\,, \qquad (4.15)
$$

$$
v(n,k,l,p) \;=\; \frac{1}{5}A_4^{-1}\begin{pmatrix} n \\ k \\ l \\ p \end{pmatrix}. \qquad (4.16)
$$

As we have already mentioned, the right hand side of (4.14) has 5 types of theta functions, which are classified by top q powers modulo $\mathbb{Z}$. In contrast to the $N = 3$ case, $\tilde{\theta}^j_{(5)}(\tau)$ ($j = 0, \ldots, 4$) is expressed by the sum of several theta functions. Now, we summarize our observations on these common properties in the following conjecture:

Conjecture 2. *For odd $N \geq 3$, the summands in Conjecture 1 are classified by the q-powers modulo $\mathbb{Z}$ as follows:*

$$
\theta^2_{A_{N-1}}(\tau) = q^{t_N}\tilde{\theta}^0_{(N)}(\tau) + q^{t_N+\frac{1}{N}}\tilde{\theta}^1_{(N)}(\tau) + \cdots + q^{t_N+\frac{N-1}{N}}\tilde{\theta}^{N-1}_{(N)}(\tau)\,, \qquad (4.17)
$$

where

$$
t_N = \frac{1}{2N^2}{}^t\rho A_{N-1}\rho = \frac{1}{24}\frac{N^2-1}{N}\,, \qquad (4.18)
$$

and each $\tilde{\theta}^j_{(N)}(\tau)$ has integral q expansions. Moreover, for $j = 0, \ldots, N-1$,

$$
q^{t_N+\frac{j}{N}}\tilde{\theta}^j_{(N)}(\tau) = a^j_1\theta^{\beta^j_1}_{(N)}(\tau) + a^j_2\theta^{\beta^j_2}_{(N)}(\tau) + \cdots + a^j_{k_j}\theta^{\beta^j_{k_j}}_{(N)}(\tau)\,, \qquad (4.19)
$$

and $\{a^j_p\}$, ($j = 0, \ldots, N-1$) satisfy

$$
\sum_{p=1}^{k_0} a^0_p = \sum_{p=1}^{k_1} a^1_p = \cdots = \sum_{p=1}^{k_{N-1}} a^{N-1}_p =: S_N\,. \qquad (4.20)
$$

The latter half of the above conjecture means that each $\tilde{\theta}^j_{(N)}(\tau)$ consists of sets of Kapranov's theta functions satisfying the property that the sum of coefficients is equal to a fixed integer S_N. For example, $S_3 = 1$ and $S_5 = 5$. Even for the $N = 7$ case, Conjecture 2 is true and it played the role of powerful guide line to find the explicit decomposition.

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$N = 7$ **case.**

$$
\begin{aligned}
\theta^2_{A_6}(\tau) = {} & \left.\begin{array}{l}
7\theta^{(2,1,1,0,0,0)}_{(7)}(\tau) + 7\theta^{(4,2,2,0,0,0)}_{(7)}(\tau) + 7\theta^{(6,3,3,0,0,0)}_{(7)}(\tau) + \\
+ 21\theta^{(1,1,1,1,1,1)}_{(7)}(\tau) + 7\theta^{(0,3,1,0,1,0)}_{(7)}(\tau) +
\end{array}\right\} =: q\tilde{\theta}^5_{(7)}(\tau) \\
& \left.\begin{array}{l}
+ \theta^{(0,1,0,0,1,0)}_{(7)}(\tau) + 2\theta^{(2,2,0,2,0,0)}_{(7)}(\tau) + 2\theta^{(5,0,1,0,0,1)}_{(7)}(\tau) + \\
+ 2\theta^{(0,5,0,1,0,0)}_{(7)}(\tau) + 2\theta^{(1,0,5,0,1,0)}_{(7)}(\tau) + 4\theta^{(0,2,2,1,0,0)}_{(7)}(\tau) + \\
+ 4\theta^{(2,0,2,2,1,0)}_{(7)}(\tau) + 4\theta^{(1,2,2,0,2,0)}_{(7)}(\tau) + 6\theta^{(1,1,1,0,3,0)}_{(7)}(\tau) + \\
+ 8\theta^{(0,1,1,1,1,0)}_{(7)}(\tau) + 14\theta^{(1,1,1,1,0,3)}_{(7)}(\tau) +
\end{array}\right\} =: q^{\frac{2}{7}}\tilde{\theta}^0_{(7)}(\tau) \\
& \left.\begin{array}{l}
+ \theta^{(0,2,0,0,2,0)}_{(7)}(\tau) + 2\theta^{(4,4,0,4,0,0)}_{(7)}(\tau) + 2\theta^{(10,0,2,0,0,2)}_{(7)}(\tau) + \\
+ 2\theta^{(0,10,0,2,0,0)}_{(7)}(\tau) + 2\theta^{(2,0,10,0,2,0)}_{(7)}(\tau) + 4\theta^{(0,4,4,2,0,0)}_{(7)}(\tau) + \\
+ 4\theta^{(4,0,4,4,2,0)}_{(7)}(\tau) + 4\theta^{(2,4,4,0,4,0)}_{(7)}(\tau) + 6\theta^{(2,2,2,0,6,0)}_{(7)}(\tau) + \\
+ 8\theta^{(0,2,2,2,2,0)}_{(7)}(\tau) + 14\theta^{(2,2,2,2,0,6)}_{(7)}(\tau) +
\end{array}\right\} =: q^{\frac{8}{7}}\tilde{\theta}^6_{(7)}(\tau) \\
& \left.\begin{array}{l}
+ \theta^{(0,3,0,0,3,0)}_{(7)}(\tau) + 2\theta^{(6,6,0,6,0,0)}_{(7)}(\tau) + 2\theta^{(15,0,3,0,0,3)}_{(7)}(\tau) + \\
+ 2\theta^{(0,15,0,3,0,0)}_{(7)}(\tau) + 2\theta^{(3,0,15,0,3,0)}_{(7)}(\tau) + 4\theta^{(0,6,6,3,0,0)}_{(7)}(\tau) + \\
+ 4\theta^{(6,0,6,6,3,0)}_{(7)}(\tau) + 4\theta^{(3,6,6,0,6,0)}_{(7)}(\tau) + 6\theta^{(3,3,3,0,9,0)}_{(7)}(\tau) + \\
+ 8\theta^{(0,3,3,3,3,0)}_{(7)}(\tau) + 14\theta^{(3,3,3,3,0,9)}_{(7)}(\tau) +
\end{array}\right\} =: q^{\frac{4}{7}}\tilde{\theta}^2_{(7)}(\tau) \\
& \left.\begin{array}{l}
+ \theta^{(0,0,1,1,0,0)}_{(7)}(\tau) + 2\theta^{(4,0,1,0,0,0)}_{(7)}(\tau) + 2\theta^{(2,4,0,1,0,0)}_{(7)}(\tau) + \\
+ 2\theta^{(1,0,0,5,0,0)}_{(7)}(\tau) + 3\theta^{(2,1,0,0,1,2)}_{(7)}(\tau) + 4\theta^{(2,0,1,1,1,0)}_{(7)}(\tau) + \\
+ 4\theta^{(2,1,2,1,0,0)}_{(7)}(\tau) + 4\theta^{(1,2,1,2,1,0)}_{(7)}(\tau) + 5\theta^{(1,0,2,2,0,1)}_{(7)}(\tau) + \\
+ 6\theta^{(1,2,1,0,0,1)}_{(7)}(\tau) + 8\theta^{(1,1,0,2,2,0)}_{(7)}(\tau) + 8\theta^{(2,2,0,1,1,1)}_{(7)}(\tau) +
\end{array}\right\} =: q^{\frac{3}{7}}\tilde{\theta}^1_{(7)}(\tau) \\
& \left.\begin{array}{l}
+ \theta^{(0,0,2,2,0,0)}_{(7)}(\tau) + 2\theta^{(8,0,2,0,0,0)}_{(7)}(\tau) + 2\theta^{(4,8,0,2,0,0)}_{(7)}(\tau) + \\
+ 2\theta^{(2,0,0,10,0,0)}_{(7)}(\tau) + 3\theta^{(4,2,0,0,2,4)}_{(7)}(\tau) + 4\theta^{(4,0,2,2,2,0)}_{(7)}(\tau) + \\
+ 4\theta^{(4,2,4,2,0,0)}_{(7)}(\tau) + 4\theta^{(2,4,2,4,2,0)}_{(7)}(\tau) + 5\theta^{(2,0,4,4,0,2)}_{(7)}(\tau) + \\
+ 6\theta^{(2,4,2,0,0,2)}_{(7)}(\tau) + 8\theta^{(2,2,0,4,4,0)}_{(7)}(\tau) + 8\theta^{(4,4,0,2,2,2)}_{(7)}(\tau) +
\end{array}\right\} =: q^{\frac{5}{7}}\tilde{\theta}^3_{(7)}(\tau) \\
& \left.\begin{array}{l}
+ \theta^{(0,0,3,3,0,0)}_{(7)}(\tau) + 2\theta^{(12,0,3,0,0,0)}_{(7)}(\tau) + 2\theta^{(6,12,0,3,0,0)}_{(7)}(\tau) + \\
+ 2\theta^{(3,0,0,15,0,0)}_{(7)}(\tau) + 3\theta^{(6,3,0,0,3,6)}_{(7)}(\tau) + 4\theta^{(6,0,3,3,3,0)}_{(7)}(\tau) + \\
+ 4\theta^{(6,3,6,3,0,0)}_{(7)}(\tau) + 4\theta^{(3,6,3,6,3,0)}_{(7)}(\tau) + 5\theta^{(3,0,6,6,0,3)}_{(7)}(\tau) + \\
+ 6\theta^{(3,6,3,0,0,3)}_{(7)}(\tau) + 8\theta^{(3,3,0,6,6,0)}_{(7)}(\tau) + 8\theta^{(6,6,0,3,3,3)}_{(7)}(\tau)
\end{array}\right\} =: q^{\frac{6}{7}}\tilde{\theta}^4_{(7)}(\tau) \,,
\end{aligned} \tag{4.21}
$$

where

$$
\theta^{(n,k,l,p,r,s)}_{(7)}(\tau) = \sum_m q^{\frac{7}{2}t(m+v(n,k,l,p,r,s))A_6(m+v(n,k,l,p,r,s))} \,, \tag{4.22}
$$

$$
v(n,k,l,p,r,s) = \frac{1}{7} A_6^{-1} \begin{pmatrix} n \\ k \\ l \\ p \\ r \\ s \end{pmatrix} . \tag{4.23}
$$

This computation was very hard because $N=7$ is critical bound for the power of our computer. However, Conjecture 2 is also true with $S_7 = 49$. Here, we have to remark on a subtle point. In this case, decomposition of $\tilde{\theta}^j_{(7)}(\tau)$ into $\theta^{\beta_j}_{(7)}(\tau)$'s is not unique because there are some non-trivial identities between $\theta^{\beta_j}_{(7)}(\tau)$'s. Up to now, we have confirmed our conjecture up to $N \leq 7$ case, but we believe that Conjecture 2 is also true for odd $N > 7$ even if N is not prime.

5. Final formulas

Finally, we generalize our previous work on $\mathrm{SU}(2)/\mathbb{Z}_2$ partition function on orbifold-$T^4/\mathbb{Z}_2$ [9] to $\mathrm{SU}(N)/\mathbb{Z}_N$ case, using the formulas prepared in section 3 and 4. We choose the following processes. First we introduce $K3$ partition function Z^{K3}, and then separate Z^{K3} into Z^{S_0} (the contribution from S_0) and Z^B (the blow-up formula). As a concrete example, we choose the partition function of $\mathrm{SU}(N)/\mathbb{Z}_N$ theory on $K3$ surface with 't Hooft flux $v^2 = 0 \pmod N$. Generalization to other cases is obviously straightforward. Using the same method as the one in section 2, the partition function is given by,

$$\begin{aligned}
Z^{K3}_0(\tau) &= \frac{1}{N}\left(\frac{1}{\eta^{24}(\frac{\tau}{N})} + \frac{1}{\eta^{24}(\frac{\tau+1}{N})} + \cdots + \frac{1}{\eta^{24}(\frac{\tau+N-1}{N})}\right) \\
&= \frac{1}{N}\left(\frac{1}{\eta^{8}(\frac{\tau}{N})}\frac{1}{\eta^{16}(\frac{\tau}{N})} + \frac{1}{\eta^{8}(\frac{\tau+1}{N})}\frac{1}{\eta^{16}(\frac{\tau+1}{N})} + \cdots + \frac{1}{\eta^{8}(\frac{\tau+N-1}{N})}\frac{1}{\eta^{16}(\frac{\tau+N-1}{N})}\right) \\
&= Z^{S_0}_0(\tau)Z^B_1(\tau) + Z^{S_0}_1(\tau)Z^B_0(\tau) + \cdots + Z^{S_0}_{N-1}(\tau)Z^B_2(\tau).
\end{aligned} \tag{5.1}$$

Here $Z^{S_0}_j(\tau)$ is already computed in section 2 as follows:

$$\begin{aligned}
Z^{S_0}_j(\tau) &= \frac{1}{N}\left(\frac{1}{\eta^{8}(\frac{\tau}{N})} + \zeta_N^{\frac{1}{3}}\zeta_N^{-j}\frac{1}{\eta^{8}(\frac{\tau+1}{N})} + \cdots + \zeta_N^{\frac{N-1}{3}}\zeta_N^{-j(N-1)}\frac{1}{\eta^{8}(\frac{\tau+N-1}{N})}\right) \\
&= q^{-\frac{1}{3N}}(q^{\frac{j}{N}} + \cdots).
\end{aligned} \tag{5.2}$$

From (5.1) and (5.2), we can pick up $Z^B_j(\tau)$, the contribution from blowing up the 16 singularities:

$$\begin{aligned}
Z^{B}_j(\tau) &= \frac{1}{N}\left(\frac{1}{\eta^{16}(\frac{\tau}{N})} + \zeta_N^{\frac{2}{3}}\zeta_N^{-j}\frac{1}{\eta^{16}(\frac{\tau+1}{N})} + \cdots + \zeta_N^{\frac{2(N-1)}{3}}\zeta_N^{-j(N-1)}\frac{1}{\eta^{16}(\frac{\tau+N-1}{N})}\right) \\
&= q^{-\frac{2}{3N}}(q^{\frac{j}{N}} + \cdots).
\end{aligned} \tag{5.3}$$

Using the key identities and Conjecture 2, we can rewrite (5.3) in terms of blow-up formulas as follows:

$$\begin{aligned}
Z^B_j(\tau) &= \frac{1}{N}\frac{1}{\eta^{16N}(\tau)}\Big((\theta^2_{A_{N-1}}(\tau))^{16} + \zeta_N^{-j}(\theta^2_{A_{N-1}}(\tau+1))^{16} + \cdots \\
&\qquad \cdots + \zeta^{-j(N-1)}(\theta^2_{A_{N-1}}(\tau+N-1))^{16}\Big) \\
&= \frac{1}{N}\frac{1}{\eta^{16N}(\tau)}q^{\frac{2}{3}\frac{N^2-1}{N}}\left[\left(\sum_{j=0}^{N-1} q^{\frac{j}{N}}\tilde{\theta}^j_{(N)}(\tau)\right)^{16}\right]_{q^{\frac{j}{N}}},
\end{aligned} \tag{5.4}$$

where $[\cdots]_{q^{\frac{j}{N}}}$ stands for the operation of projecting out $q^{\frac{j}{N}+n}$ $(n \in \mathbb{Z}_{\geq 0})$ powers. Here we will give explicit expression of (5.4) in the $N = 3$ case:

$$\begin{aligned}
Z_0^B(\tau) &= \frac{1}{3}\frac{1}{\eta^{48}(\tau)}\left((\theta_{A_2}^2(\tau))^{16} + (\theta_{A_2}^2(\tau+1))^{16} + (\theta_{A_2}^2(\tau+2))^{16}\right) \\
&= \frac{1}{3}\frac{1}{\eta^{48}(\tau)} q^{\frac{16}{9}} \left[\left(\tilde{\theta}_{(3)}^0(\tau) + q^{\frac{1}{3}}\tilde{\theta}_{(3)}^1(\tau) + q^{\frac{2}{3}}\tilde{\theta}_{(3)}^2(\tau)\right)^{16}\right]_{q^{\frac{0}{3}}} \\
&= \frac{1}{3}\frac{1}{\eta^{48}(\tau)} q^{\frac{16}{9}} \left((\tilde{\theta}_{(3)}^0(\tau))^{16} + 560q(\tilde{\theta}_{(3)}^0(\tau))^{13}(\tilde{\theta}_{(3)}^1(\tau))^3 + \cdots\right. \\
&\qquad\qquad \left.\cdots + 16q^{10}(\tilde{\theta}_{(3)}^0(\tau))(\tilde{\theta}_{(3)}^2(\tau))^{15}\right). \qquad (5.5)
\end{aligned}$$

In this way, we have reconstructed $K3$ partition function which has the factor of the partition function on S_0 and that of the blow-up formulas. Note that our processes in this section are opposite direction to our previous work [9]. In the previous work on SU(2) gauge group, we first prepared the partition function on S_0 and $\mathcal{O}(-2)$ curve blow-up formulas. Next, we multiplied and summed up these factors so that the total partition function satisfy the S-duality conjecture. The final result was surely the $K3$ partition function derived by Vafa and Witten. In both cases, the expression of the partition function we have obtained has the same structure, i.e., it consists of the factor of the partition function on S_0 and that of the blow-up formulas. But we have to notice that the contribution from the blow-up formulas can be expressed by $\mathcal{O}(-N)$ blow-up formulas with $j/N + \mathbb{Z}_{\geq 0}$ powers. Of course, this result still needs appropriate geometrical interpretation.

6. Conclusion

In this paper, we reconstructed the $K3$ partition function of $\mathcal{N} = 4$ $\mathrm{SU}(N)/\mathbb{Z}_N$ gauge theory using orbifold construction $T^4/\mathbb{Z}_2$. The key point lies on the fact that we can rewrite the function $1/\eta(\tau/N)$ in terms of $\mathcal{O}(-N)$ curve blow-up formulas.

The remaining most serious problem is geometrical interpretation of $\mathcal{O}(-N)$ curve blow-up formulas in our theory. Since coefficients of $\mathcal{O}(-N)$ curve blow-up formulas are all integers, we hope that there may be some nice geometrical interpretation similar to the one in the SU(2) case.

From the point of view of the work of Nakajima [21], we have to say that, for the SU(N) case, the contribution from blow-up comes from U(N) gauge theory on ALE A_{N-1} space. Then appearance of level N SU(N) theta functions seems to be natural. If we believe these scenario, we can expect the following interesting applications of our result. According to [21], there are already ADE gauge theory on ALE space associated with the same ADE Lie algebra. Then, we can speculate the form of "ADE blow-up" formula using the denominator identity of ADE affine Lie algebra as we did in section 3 of this paper. Moreover, we may be able to determine the form of the partition function of $\mathcal{N} = 4$ ADE gauge theory on $K3$ surfaces, by tracing back the process of our computation in this paper.

The most crucial point in this paper is the process of rewriting the function $1/\eta(\tau/N)$ in terms of $\mathcal{O}(-N)$ curve blow-up formula. This process seems to suggest existence of a kind of duality between the bulk contribution and the blow-up contribution in the sense of Vafa and Witten [2]. Physical meaning of this duality remains a problem to be cleared up. We expect that we may estimate the form of the partition function of ADE gauge theory on $K3$, if we apply the formula to the "ADE blow-up formula".

Finally, we point out level-rank duality in affine Lie algebra, that states duality between $\hat{sl}(l)_r$ and $\hat{sl}(r)_l$ [26]. In our case, we observed appearance of $\hat{sl}(N)_N$, which is self-dual in the context of level-rank duality. We expect that this special symmetry may have some physical meaning.

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